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Pointwise Saturation for Variation Diminishing Convolution Transforms

R. J. NESSEL AND R. L. STENS

Lehrstuhl A für Mathematik, Rheinisch-Westfälische Technische Hochschule Aachen, 51 Aachen, West Germany

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As an application of a general result for positive operators due to Lorentz-Schumaker [7], we consider pointwise saturation for an important class of operators arising in connection with the real inversion formula of variation diminishing convolution transforms, a field initiated and studied extensively by Schoenberg [8] and Hirschman and Widder [6]. The operators are given by

$$K_n(f; x) = \int_{-\infty}^{\infty} G_n(x - t) f(t)\, dt,$$
$$G_n(t) = (1/2\pi i) \int_{-i\infty}^{i\infty} E_n(s) e^{st}\, ds, \quad E_n(s) = e^{b_n s} \prod_{k=n+1}^{\infty} (1 - s/a_k) e^{s/a_k}, \tag{1}$$

with real b_n and a_k satisfying $b_n = o(n)$, $n \to \infty$, and $\sum_{k=1}^{\infty} a_k^{-2} < \infty$. Some fundamental properties (cf. [3; 4; 6, pp. 56, 125]) are,

(i) $G_n(x) \geqslant 0$,

(ii) $\| G_n \|_{L^1} = 1$,

(iii) $\int_{-\infty}^{\infty} x G_n(x)\, dx = b_n$, (2)

(iv) $\int_{-\infty}^{\infty} x^2 G_n(x)\, dx = \sigma_n + b_n^2 \quad$ with $\quad \sigma_n = \sum_{k=n+1}^{\infty} a_k^{-2}$,

(v) $G_n(t) \leqslant K \sigma_n^{-1/2} \exp\{-\frac{1}{4} \sigma_n^{-1/2} \mid t - b_n \mid\}$, which, in particular, implies, for any $\delta > 0$,

$$\text{(v}'\text{)}\quad \int_{|t|>\delta} t^2 G_n(t)\,dt = o(\sigma_n).$$

Here $L^p = L^p(-\infty, \infty)$, $1 \leqslant p \leqslant \infty$, are the usual Lebesgue spaces whereas $C(-\infty, \infty)$ is the set of all functions, continuous and bounded on $(-\infty, \infty)$.

The norm-saturation theorem for (1) was given by Ditzian [3] (see also [4, 5]). For pointwise saturation one may essentially use the same techniques.

PROPOSITION. *Let* $\{\chi_n\} \subset L^1$ *be such that for some null-sequence* $\{\mu_n\}$, $\mu_n \neq 0$, *and for constants* $\gamma_1, \gamma_2 \in \mathbf{R}$,

$$\begin{aligned} &\text{(i)} && \int_{-\infty}^{\infty} \chi_n(u)\,du = 1 \\ &\text{(ii)} && \int_{-\infty}^{\infty} u\chi_n(u)\,du = \gamma_1 \cdot \mu_n + o(\mu_n), \\ &\text{(iii)} && \int_{-\infty}^{\infty} u^2\,|\,\chi_n(u)|\,du = O(\mu_n), \\ &\text{(iv)} && \int_{-\infty}^{\infty} u^2\chi_n(u)\,du = \gamma_2\mu_n + o(\mu_n), \\ &\text{(v)} && \int_{|u|\geqslant\delta} u^2\,|\,\chi_n(u)|\,du = o(\mu_n), \qquad \text{for each} \quad \delta > 0. \end{aligned} \tag{3}$$

If $f \in L^\infty$, *the following Voronowskaya-type relation holds*:

$$\lim_{n\to\infty} \frac{1}{\mu_n} \int_{-\infty}^{\infty} [f(x-u) - f(x)]\chi_n(u)\,du = -\gamma_1 f'(x) + (\gamma_2/2) f''(x),$$

whenever $f''(x)$ *exists*.

Following classical lines (cf. [4; 2, p. 138]) the proof is straightforward. Indeed,

$$\begin{aligned} \int_{-\infty}^{\infty} [f(x-u) - f(x)]\chi_n(u)\,du &= \int_{-\infty}^{\infty} [f(x-u) - f(x) + uf'(x)]\chi_n(u)\,du \\ &\quad - \int_{-\infty}^{\infty} f'(x)\,u\chi_n(u)\,du \equiv I_1 + I_2 \end{aligned}$$

so that $I_2 = -\gamma_1 f'(x)\,\mu_n + o(\mu_n)$ by (3)(ii). Since $f''(x)$ exists,

$$f(x-u) - f(x) + uf'(x) = (\tfrac{1}{2}) f''(x) \cdot u^2 + \eta(u) \cdot u^2,$$

with some $\eta \in L^\infty$ satisfying $\lim_{u\to 0} \eta(u) = 0$. Therefore, by (3)(iii), (v) $\int_{-\infty}^{\infty} | u^2\eta(u) \chi_n(u)| \, du = o(\mu_n)$, and it follows by (3)(iv) that

$$I_1 = (\gamma_2/2) f''(x) \mu_n + o(\mu_n),$$

which proves the assertion.

Obviously, if $b_n = (\alpha/2) \sigma_n + o(\sigma_n)$ for some $\alpha \in \mathbf{R}$, the sequence $\{G_n\}$ (cf. (1)) satisfies (3) with $\mu_n = \sigma_n$, $\gamma_1 = \alpha/2$, $\gamma_2 = 1$.

Now one may apply a general pointwise saturation theorem for positive linear operators satisfying Voronowskaya-type conditions (see [1, 7] and the literature cited therein). Thus, with $\rho(x) = \frac{1}{2}$, $\phi(x) = \int_a^x e^{\alpha t} \, dt$, $D = -\alpha(d/dx) + (d/dx)^2$ in the terminology of [1], Theorem 2,

COROLLARY. *Let $f \in C(-\infty, \infty)$ and let K_n be given by* (1) *such that $b_n = (\alpha/2) \sigma_n + o(\sigma_n)$ for some $\alpha \in \mathbf{R}$.*

(a) *If* $\liminf_{n\to\infty} \sigma_n^{-1}[K_n(f; x) - f(x)] = 0$ *for each $x \in (a, b)$, then for all $x, x_0, x_1 \in [a, b]$, $x_0 \leq x \leq x_1$,*

$$f(x) = \frac{\phi(x) - \phi(x_0)}{\phi(x_1) - \phi(x_0)} f(x_1) + \frac{\phi(x_1) - \phi(x)}{\phi(x_1) - \phi(x_0)} f(x_0), \; \left(\phi(x) = \int_a^x e^{\alpha t} \, dt\right).$$

(b) *If $\sigma_n^{-1} | K_n(f; x) - f(x)| \leq M/2 + o_x(1)$ for each $x \in (a, b)$, then $f'(x)$ exists for all $x \in [a, b]$, belongs to $C[a, b]$, and*

$$| e^{-\alpha x} f'(x) - e^{-\alpha y} f'(y)| \leq M \left| \int_x^y e^{-\alpha t} \, dt \right|,$$

for all $x, y \in [a, b]$, and vice versa.

Obviously, in the particular case $\alpha = 0$, the assertion of (a) states that f is linear on $[a, b]$, whereas (b) delivers $f' \in \mathrm{Lip}_M\, 1$.

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